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Brief communication

Pressure loads on a deforming body moving in a weakly stratified inviscid fluid near boundaries

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1. Introduction

One of the key problems in multiphase flows is that of predicting the high-Reynolds number motion of bubbles or bubbly flows in a non-uniform ambient flow. It is well known that in the range of moderate to large Reynolds number of clean bubbles, inviscid induced inertia effects may be decoupled from the viscous ones. Depending on the relative importance of surfactant effects and bubble size, the bubble can be considered as a rigid or deformable sphere. It is also evident that large bubbles tend to loose their sphericity and become elongated (spheroidal-like shapes). The motion in a bubble cloud, even in a dispersed one, is very complex also due to the non-homogeneity of the ambient flow. Non-homogeneous effects can be induced by the proximity of other moving bodies causing flow non-uniformities (which can be modeled within the inviscid irrotational realm), shear induced forces (mainly of a lift type) generated by the ambient vorticity field (assumed to be uniform) or by the baroclinical induced lift and drag forces resulting from the weak density stratification of the ambient fluid. Determining the equations of motion for bubbles by combining all of the above mentioned effects, is indeed a formidable task, and many path instability phenomena observed in a cloud, such as zig-zagging, spiralling or bubble coalescence, are still not well understood. In this context, a reference should be made to the comprehensive recent review article by Magnaudet and Eames (2000), hereafter denoted as ME, which served as a motivation for studying the titled problem.

An elegant and useful result presented in ME (Eq. (9)) is that for the force acting on a *rigid* particle with volume $\forall$ translating steadily with a linear velocity $\vec{U}$ in an inviscid fluid of infinite expanse with a weak uniform density gradient $\nabla\rho$:

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$$\vec{F} = -C_{\mathrm{D}_\rho}\forall(\vec{U}\cdot\nabla\rho)\vec{U} + C_{\mathrm{L}_\rho}\forall(\vec{U}\times\nabla\rho)\times\vec{U}, \tag{1}$$

where the baroclinic drag coefficient C_{D_ρ} and lift coefficient C_{L_ρ}, are equal to half the added-mass coefficient C_{M}, i.e.,

$$C_{\mathrm{D}_\rho} = C_{\mathrm{L}_\rho} = C_{\mathrm{M}}/2. \tag{2}$$

The acclaimed D'Alembert paradox in inviscid hydrodynamics indicates that the pressure force acting on a rigid body moving steadily with a rectilinear velocity in an unbounded homogeneous quiescent fluid is null. If on the other hand, the ambient fluid has a uniform weak density gradient, the same particle exhibits lift and drag forces, depending on the relative orientation between the density gradient and the particle velocity. These forces depend linearly on the ambient density gradient and quadratically on the body velocities. Eq. (1) is given in ME without proof and is claimed to be valid only for an axisymmetric particle moving along its major axis of symmetry. For a detailed proof, the reader is also referred to Eames and Hunt (1997), who examined the steady non-Boussinesq flow of a rigid body in a stratified medium, under the assumption that the characteristic length-scale of the density non-uniformity is much larger than the characteristic length of the particle. Lagrangian formulations and the concept of the Darwin drift function (Lighthill, 1956) are invoked to calculate the rotational flow associated with the displacement of the isopycnal surfaces advected by the irrotational component of the flow field. An analogy between a steady flow in a direction perpendicular to a density gradient and that in a linear shear flow (Yih, 1959), is also applied for the particular spherical configuration. A final expression for the force is then obtained by performing a momentum flux balance over a large control surface surrounding the body, by assuming that the fluid domain is unbounded.

In spite of the relative simple form of the baroclinic force expression (1), its derivation following Eames and Hunt (1997) is rather involved and is based on several physical assumptions i.e., steady motion, unbounded fluid and rigid body. In addition, it may be valid only for spherical shapes or other axisymmetric restricted shapes. Moreover, it does not unveil the possible important contributions of the angular velocity components on the total force nor does it expound the corresponding expression for the torque acting on general non-spherical particles moving with six degrees of freedom, (i.e., translation and rotation). Needless to say, that path trajectories of non-spherical large bubbles are determined from the numerical solution of the six corresponding equations of motion for the force and torque acting on the particle. The lack of a concise derivation of the torque expression for a high Reynolds non-spherical bubble embedded in a non-homogeneous medium, is indeed the first in the list of three important open problems in bubble dynamics (see ME, p. 701).

A different approach based on the classical Hamiltonian mechanics, has been also recently applied by Palierne (1999) for the same problem, by treating the body plus fluid as a combined single dynamical system. A rigid body of arbitrary shape which moves with six velocity components in a quiescent unbounded fluid with a constant density gradient is assumed. An asymptotic theory is then constructed, where the small parameter is the ratio of the effective diameter of the body to the length-scale of the density non-uniformity. The zeroth-order solution is assumed to be governed by the potential flow field and vorticity effects, such as vortex stretching or the evolution of the ambient vorticity (also due to stratification), are ignored altogether. Palierne (1999) derived a corresponding expression for the torque acting on a rigid moving particle which is missing in

Eames and Hunt (1997). In addition, EH's derivation for the force (see their Table 1), incorrectly distinguishes between 2-D and 3-D shapes, whereas that of Palierne does not discriminate between the two. Palierne's expressions do not exhibit any immediate resemblance to the rather simple force expression (1) which clearly display, in a vector form, the drag and lift components.

One of the purposes of the present note is to provide a generalization of the above mentioned analyses, so as to account for deformable bodies moving unsteadily in the close proximity of other bodies or fixed boundaries. This alternative proof, can be then compared against that of Eames and Hunt and Palierne and provide the missing linkage between the two. In fact, it is shown that the corresponding expressions for the force and moment acting on an arbitrary particle moving in a weak stratification, are readily obtained as a limiting case of a well-known general theorem extending the Kelvin–Kirchhoff approach to account for a well-known dependence on generalized coordinates (Miloh and Landweber, 1981). In addition to yielding a rather trivial solution for the unbounded stratified case, the present methodology can be easily applied for non-rigid shapes and for analyzing wall effects or flow interfaces. In some situations (as demonstrated in the sequel) the baroclinic force induced by the flow stratification, can act in the opposite direction to the hydrodynamic inertia force induced by a nearby boundary, so as to yield a vanishing small force. The structure of this note is as follows: first we present the generalized so-called Kelvin–Kirchhoff expressions for the pressure loads for deformable arbitrary shapes moving unsteadily in a bounded weakly stratified inviscid fluid. The solutions of EH and Palierne are obtained as limiting cases and the connection between the two is established. The last two sections include demonstrations for both rigid and pulsating bubbles moving near a rigid wall in cases where the baroclinic forces act in the opposite direction to the inertia ones.

2. The generalized Kelvin–Kirchhoff equations

Let us consider a 3-D rigid or deformable body moving unsteadily with linear $\vec{U}(U_1, U_2, U_3)$ and angular $\vec{\Omega}(U_4, U_5, U_6)$ velocities in an otherwise quiescent bounded incompressible fluid. We employ two rectangular Cartesian coordinate systems. One is X_i, where $i = 1, 2, 3$, fixed in space and the other x_i is attached to the moving body with the origin taken at its mass centroid. The instantaneous position and orientation of the body are defined by the six generalized coordinates, namely the coordinates of the origin X_{0i} and by three independent angles θ_i, such as the Euler angles, which define the unitary transformation matrix $C_{ij}(\theta_1, \theta_2, \theta_3)$ between the fixed X_i and moving x_i systems. Thus, the rectilinear velocity components can be expressed in terms of the generalized velocities $\dot{X}_{0i}$ as

$$U_i = C_{ij}\dot{X}_{0j}, \quad \dot{X}_{0i} = C_{ji}U_j, \tag{3}$$

where the dot denotes differentiation with respect to time. In a similar manner, the angular velocity components U_{i+3}, can be expressed in terms of $\dot{\theta}_i$, the time rate of change of θ_i, as

$$U_{i+3} = D_{ij}\dot{\theta}_j, \quad \dot{\theta}_i = D^{\mathrm{I}}_{ij}U_{j+3}, \tag{4}$$

where D_{ij} is the rotational transformation matrix and D^{I}_{ij} is its inverse. A useful relation between the matrices C_{ij} and D_{ij} is (Miloh and Landweber, 1981).

$$D_{ij} = \frac{1}{2}\epsilon_{irs}\frac{\partial C_{\rm rt}}{\partial \theta_j}C_{\rm st}, \tag{5}$$

where ϵ_{irs} is the permutation tensor and all indices in (5) are equal to 1,2, or 3.

Next we express the total kinetic energy of the fluid, bounded internally by the moving deformable body and externally by the presence of some fixed boundaries, as

$$2T = A_{\alpha\beta}(X_i,\theta_j,t)U_\alpha U_\beta + 2A_{\alpha d}(X_i,\theta_j,t)U_\alpha + A_{\rm dd}(X_i,\theta_j,t), \quad \alpha,\beta = 1,2,\ldots,6 \tag{6}$$

which depend quadratically on the body's six velocities U_α. Here $A_{\alpha\beta}$ is the traditional symmetric second-order added-mass tensor, $A_{\alpha\rm d}$ is the deformation vector describing the interaction between the body deformation and its rigid body motion and $A_{\rm dd}$ is a scalar representing pure deformation. These added-mass like coefficients generally depend on the generalized coordinates and time. If the body is non-deformable (rigid), then both $A_{\alpha\rm d}$ and $A_{\rm dd}$ vanish and the remaining added-mass tensor $A_{\alpha\beta}$ does not explicitly depend on time. In the case of a rigid body moving in an infinite expanse of a quiescent fluid, the $A_{\alpha\beta}$'s are related to the displaced mass of the fluid by some coefficients depending only on the body's shape. However, if a rigid body is moving in a confined space (i.e., in the proximity of other bodies or near boundaries), then the added-mass tensor in (6) depends on the generalized coordinates X_i and θ_j which describe the instantaneous location and orientation of the body with respect to the fixed system.

Let us next use the general Lagrangian energy approach formulated in Miloh and Landweber (1981), in conjunction with (6). This procedure leads to the following analytic expressions for the force F_i and moment M_i components exerted by the inviscid fluid on the moving deformable body:

$$F_i = -\dot{p}_i + \epsilon_{ijk}p_jU_{k+3} - \frac{\partial p_i}{\partial X_j}C_{kj}U_k - \frac{\partial p_i}{\partial \theta_j}D^{\rm I}_{jk}U_{k+3} + \frac{\partial T}{\partial X_j}C_{ij} \tag{7}$$

and

$$M_i = -\dot{q}_i + \epsilon_{ijk}p_jU_k + \epsilon_{ijk}q_jU_{k+3} - \frac{\partial q_i}{\partial X_j}C_{kj}U_k - \frac{\partial q_i}{\partial \theta_j}D^{\rm I}_{jk}U_{k+3} + \frac{\partial T}{\partial \theta_j}D^{\rm I}_{ji}, \tag{8}$$

where again $i,j,k = 1,2,3$ and $\alpha,\beta = 1,2,\ldots,6$. In the above we have defined the generalized hydrodynamic impulse $\vec{p}$ and angular impulse $\vec{q}$ by,

$$p_i = A_{\alpha i}U_\alpha + A_{{\rm d}i}, \quad q_i = A_{\alpha,i+3}U_\alpha + A_{{\rm d},i+3} \tag{9}$$

Eqs. (7) and (8) can be thus considered as a generalization of the corresponding *rigid* body analyses of Miloh and Landweber for the case of deformable non-rigid bodies.

Several special cases can be next discussed:

Case 1: A rigid body moving in an unbounded domain of homogeneous and inviscid fluid in an irrotational motion. The only surviving term here is the first quadratic term in the r.h.s of (6) where the $A_{\alpha\beta}$'s are constants. For this case the velocity field $\vec{v}$ can be determined from a velocity potential function, i.e., $\vec{v} = \nabla\phi$, where $\phi = U_\alpha\varphi_\alpha$ and φ_α are the six unit Kirchhoff potentials satisfying the following Neumann type boundary condition on the body surface S, i.e., $(\partial\phi_\alpha/\partial n)|_s = n_\alpha$ and a proper decay condition at infinity. Here n_i denotes the three components of the normal to the body vector $\vec{n}$ directed into the fluid and n_{i+3} are the three components of the cross product $\vec{r}\times\vec{n}$ between the body radius vector $\vec{r}$ (from the body's centroid to a point on the surface) and $\vec{n}$. Using the Gauss theorem implies that the added-mass tensor $A_{\alpha\beta}$, defining the kinetic energy of the fluid (6), is a purely geometrical constant given by

$$A_{\alpha\beta} = -\rho \int_S \varphi_\alpha n_\beta \, \mathrm{d}s = -\rho \int_S \varphi_\beta n_\alpha \, \mathrm{d}s = A_{\beta\alpha}, \tag{10}$$

where ρ is the uniform ambient density of the fluid. Thus, the last three terms in both (7) and (8), which involve partial derivatives of $A_{\alpha\beta}$ vanish, and one readily recovers the classical (unbounded case) Kelvin–Kirchhoff expressions for the force and moment in terms of the added-mass coefficients.

Case 2: A deformable body moving in an unbounded flow domain of inviscid fluid with uniform density. The total velocity potential is given now by $\phi = U_\alpha \varphi_\alpha + \varphi_\mathrm{d}$, where φ_d is the so-called deformation potential. If the time dependent shape of the deformable body is denoted by $S(\vec{r}, t) = 0$, then the Neumann type boundary condition for φ_d is related to the instantaneous shape of the body by $(\partial \varphi_\mathrm{d}/\partial n)|_S = -\dot{S}/|\nabla S|$, where the dot represents differentiation with respect to time. Here all the added-mass like terms defining the total kinetic energy of the fluid depend explicitly on time, since S does. The various deformation terms appearing in (6) can be thus defined as:

$$A_{\alpha\mathrm{d}} = -\rho \int_S \varphi_\mathrm{d} n_\alpha \, \mathrm{d}S = A_{\mathrm{d}\alpha}, \quad A_{\mathrm{dd}} = -\rho \int_S \varphi_\mathrm{d} \frac{\partial \varphi_\mathrm{d}}{\partial n} \, \mathrm{d}S = \rho \int_S \varphi_\mathrm{d} \frac{\dot{S}}{|\nabla S|} \, \mathrm{d}S. \tag{11}$$

For the case of a deformable body in an infinite fluid the term $A_{\alpha\mathrm{d}}$, can be identified as the Kelvin impulse ($\alpha = 1, 2, 3$) or the Kelvin impulse-couple ($\alpha = 4, 5, 6$). For further definitions, including a reference to the case of self propulsion of a deformable body in an unbounded fluid, see also Miloh and Galper (1993).

Case 3: A rigid body moving in a bounded domain by some fixed boundaries of homogeneous and inviscid fluid. The external boundaries are either rigid ($\partial\phi/\partial n = 0$) or equi-potential ($\phi = 0$), modeling for example a free-surface under the high-frequency limit. It is evident that also in this case the added-mass tensor is given by (10) with the only exception that now the $A_{\alpha\beta}$'s are no longer constant and instead depend on the generalized coordinates (X_i, θ_j) describing the instantaneous position and orientation of the moving body with respect to the inertial system. The generalized expressions for the force and moment acting on the moving body in the presence of nearby boundaries, are then given by (7) and (8) respectively.

Case 4: A rigid body moving in an unbounded domain of a weakly stratified incompressible and inviscid fluid such that $a\nabla\rho/\rho \ll 1$. Here $\nabla\rho$ denotes the uniform density gradient, a represents a typical size of the body and ρ is again the ambient fluid density. Under these assumptions, the added-mass coefficients (correct to leading-order) can still be expressed by (10), but since ρ is considered now as a weak function of X_i, one simply gets

$$\frac{\partial A_{\alpha\beta}}{\partial X_j} = -\frac{\partial \rho}{\partial X_j} \int_S \varphi_\alpha n_\beta \, \mathrm{d}s = A_{\alpha\beta} \frac{\partial}{\partial X_j} (\ln \rho); \quad \frac{\partial A_{\alpha\beta}}{\partial \theta_j} = 0. \tag{12}$$

Thus, the extra dynamic terms in the equations of motion, which depend linearly on the density gradient of the surrounding fluid, are readily found from (7), (8) and (12) as

$$F_i^{(\rho)} = -\left(A_{\alpha i} C_{kj} U_k - \frac{1}{2} A_{\alpha\beta} C_{ij} U_\beta\right) U_\alpha \frac{\partial}{\partial X_j} (\ln \rho), \tag{13}$$

$$M_i^{(\rho)} = -A_{\alpha,i+3} C_{kj} U_\alpha U_k \frac{\partial}{\partial X_j} (\ln \rho). \tag{14}$$

If instead of expressing the density gradient in the laboratory (inertial) coordinate system $\partial\rho/\partial X_j$, we choose to express it in the moving (attached) system $(\partial\rho/\partial x_j) \equiv \nabla_j \rho$, where $(\partial\rho/\partial x_j) = C_{ij}(\partial\rho/\partial X_j)$, one gets

$$F_i^{(\rho)} = -\frac{1}{\rho}\left(A_{\alpha i} \nabla_k \rho U_k - \frac{1}{2} A_{\alpha\beta} \nabla_i \rho U_\beta\right) U_\alpha \tag{15}$$

$$M_i^{(\rho)} = -\frac{1}{\rho} A_{\alpha,i+3} \nabla_k \rho U_k U_\alpha \tag{16}$$

These are basically the rigid-body expressions given in Eq. (20) of Palierne (1999) using tensor notations. In order to examine the connection between the above and the corresponding analysis of Eames and Hunt (1997), it is convenient to use vector notations and express (15) and (16) in terms of the rigid impulse $\vec{p}_{\mathrm{r}}$ and angular impulse $\vec{q}_{\mathrm{r}}$ (defined in the r.h.s of Eq. (9) for $A_{\mathrm{d}\alpha} = 0$) as,

$$\begin{aligned} \vec{F}^{(\rho)} &= \frac{1}{2\rho}(\vec{p}_{\mathrm{r}} \cdot \vec{U} + \bar{q}_{\mathrm{r}} \cdot \vec{\Omega}) \nabla \rho - \frac{1}{\rho}(\vec{U} \cdot \nabla \rho) \vec{p}_{\mathrm{r}} \\ &= -\frac{1}{2\rho}(\vec{U} \cdot \nabla \rho) \vec{p}_{\mathrm{r}} + \frac{1}{2\rho}(\vec{p}_{\mathrm{r}} \times \nabla \rho) \times \vec{U} + \frac{1}{2\rho}(\vec{q}_{\mathrm{r}} \cdot \vec{\Omega}) \nabla \rho, \end{aligned} \tag{17}$$

$$\vec{M}^{(\rho)} = -\frac{1}{\rho}(\bar{U} \cdot \nabla \rho) \vec{q}_{\mathrm{r}}. \tag{18}$$

Eqs. (17) and (18) are the sought expressions for the baroclinic force and moment (terms proportional to $\nabla\rho$) acting on 2-D or 3-D translating and rotating rigid bodies.

In the case of a non-rotating ($\vec{\Omega} = 0$) axisymmetric body moving along its axis of symmetry, the added-mass tensor is purely diagonal and thus $\vec{p} = A\vec{U}$ where A is the corresponding added-mass coefficient. Thus, Eq. (17) reduces precisely to Eqs. (1) and (2) (given originally in Eq. (6.9) of EH), where $C_{\mathrm{M}} = (A/\forall)$. It is also important to note that Eq. (18) provides the corresponding closure equation for the moment from which the actual trajectory of a general body moving in a weakly stratified flow field can be obtained. Consequently, we have demonstrated the connection between the two independent analyses of Palierne and EH for the baroclinic force and also provide a generalization of the latter for arbitrary deformable shapes and motions (i.e., including body rotation). It is interesting to note that as long as the body's linear velocity is perpendicular to the gradient density, i.e., $\bar{U} \cdot \nabla\rho = 0$, the torque on the body is null regardless of the body's angular velocity. The body will always experience a lift force in a direction perpendicular to both $\vec{U}$ and $\vec{p} \times \nabla\rho$, as long as the linear impulse is not colinear with the density gradient. There exists an additional force component in the direction of $\nabla\rho$ for a rotating body, if the angular impulse $\vec{q}$ has a non-vanishing projection along the vector of angular velocity $\vec{\Omega}$.

Case 5: Finally, we consider the case of a deformable body embedded in an unbounded fluid. Here the baroclinic force is given by (17), which must be augmented by the additional term

$\frac{1}{2}A_{\mathrm{dd}}\nabla\rho$ (7), where $\vec{p}_{\mathrm{r}}$ is replaced by $\vec{p}$ (9). In a similar manner, the combined baroclinic moment is simply given by (18) when $\vec{q}_{\mathrm{r}}$ is replaced by $\vec{q}$ defined in (9).

What remains now is to find the condition under which Eqs. (17) and (18) are valid from the physical point of view. It is clear that the motion of an inviscid and stratified fluid is rotational in general, since the Lagrangian time evolution of the vorticity vector $\vec{\omega} = \nabla \times \vec{v}$ is given by the Helmholtz equation, i.e.,

$$\frac{\mathrm{D}\vec{\omega}}{\mathrm{D}t} = (\vec{\omega}\cdot\nabla)\vec{v} + (\nabla\rho \times \nabla P)/\rho^2, \tag{19}$$

where P is the dynamic pressure in the fluid. The first term on the right hand side of (19) represents 3-D vortex stretching and the second term is the so-called Bjerknes contribution. Thus, even if the motion starts from rest, say at time $t = 0^+$, where initially the ambient vorticity is null, the time derivative of the vorticity at $t = 0^+$ is non-zero. By invoking the Helmholtz decomposition for the velocity field $\vec{v} = \nabla\phi + \vec{v}_{\mathrm{r}}$, where $\vec{v}_{\mathrm{r}}$ denotes the rotational (vortical) part of $\vec{v}$ ($\nabla \times \vec{v}_{\mathrm{r}} \neq 0$), one gets therefore that the term $\vec{v}_{\mathrm{r}}$ can be ignored in the Euler equation for small time after the motion has commenced, but the inertial term $(\partial\vec{v}_{\mathrm{r}}/\partial t)$ has to be kept in the same equation, even initially. However, since the impermeable body boundary conditions are satisfied by the potential part of $\vec{v}$, one gets that $\vec{v}_{\mathrm{r}}\cdot\vec{n}|_s = 0$ and $(\partial\vec{v}_{\mathrm{r}}/\partial t)\cdot\bar{n}|_s = 0$ at $t = 0^+$. It can be shown then that the term $(\partial\vec{v}_{\mathrm{r}}/\partial t)$ in the Euler equation does not contribute to the force and moment acting on the body, since $\vec{v}_{\mathrm{r}}$ is solenoid and

$$\int_{\forall^+}\frac{\partial\vec{v}_{\mathrm{r}}}{\partial t}\cdot\nabla\varphi_\alpha \mathrm{d}\forall = \int_{\forall^+}\nabla\cdot\left(\varphi_\alpha\frac{\partial\vec{v}_{\mathrm{r}}}{\partial t}\right)\mathrm{d}\forall = -\int_S\frac{\partial\vec{v}_{\mathrm{r}}}{\partial t}\cdot\vec{n}\,\mathrm{d}s = 0, \tag{20}$$

where $\forall^+$ denotes the volume of the fluid exterior to the body.

It would be also useful to have at least an order of magnitude for the "short time interval" during which the present analysis can be used. For this purpose let us define V and l as the characteristic velocity and length scale of the body and thus denote the tilded dimensionless time by $t^* = tV/l$. The Bjerknes term in the Helmholtz vorticity transport equation, then suggests that the analysis is valid for $t^* < \rho/(l\nabla\rho)$. Clearly for inviscid homogeneous ($\rho = \mathrm{const}$) fluids, there is no restriction on time and the above derivation is valid for all time. However, for a stratified flow ($\nabla\rho \neq 0$), the time interval is inversely proportional to the ambient density gradient and the body's size.

One may conclude therefore that expressions (17) and (18) for the baroclinic force and torque acting on a body moving in a weakly stratified fluid, which have been derived using a potential framework, are still valid during a short time interval after introducing the body impulsively into an otherwise quiescent fluid medium with a uniform density gradient.

3. Baroclinic motion of a bubble near a wall

In order to demonstrate the preceding analysis, we consider the following simplified case which is connected to some applications of bubble dynamics in non-homogeneous flows. A spherical bubble of (constant) radius a is moving with a constant velocity U towards (Case A) or parallel

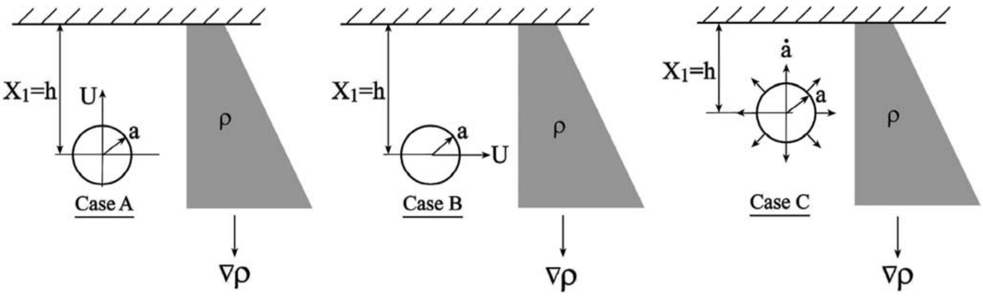

Fig. 1. Definition sketch.

(Case B) to a rigid plane wall (Fig. 1). The fluid is assumed to be inviscid and incompressible (thus, possible near-wall viscous effects are excluded) but there is a (small) uniform density gradient $\nabla\rho$ in the direction perpendicular to the wall. Such a gradient may be caused in the fluid, for example, by buoyancy effects resulting from some thermal boundary conditions (heating/cooling) applied on the wall. Effects of density stratification may then be predominant near the boundary. Because of the assumptions of flow steadiness, perfect geometrical sphericity and the neglect of viscous stresses, the D'Alembert paradox implies that once moving in an unbounded and homogeneous flow, a spherical bubble (or drop) will experience no force. However, in the presence of weak stratification, there is a drag/thrust force acting on the bubble in the opposite direction to $\nabla\rho$ and a lift force in the direction of $\nabla\rho$ (see Eq. (17)). Thus, such a baroclinic type force tends to push the bubble towards the wall in Case A and away from the wall in Case B. Recalling that the added-mass for a sphere is $A = \frac{1}{2}\rho\forall$ ($\forall$ being its volume), it follows from (17) that the baroclinic forces in both cases are

$$F_{\rm a}^{(\rho)} = -\frac{1}{4}\forall|\nabla\rho|U^2; \quad F_{\rm b}^{(\rho)} = \frac{1}{4}\forall|\nabla\rho|U^2. \tag{21}$$

However, if the bubble is moving in the proximity of a rigid wall, there exists an additional hydrodynamic force (wall effect) that may act in opposite directions to the baroclinic forces. Under some circumstances, there can be also a mutual cancellation between these two types of forces. The wall induced inertia forces are again of attraction/repulsion (from the wall) type and depend directly on the instantaneous distance h of the bubble from the rigid wall. It follows from (7) that these wall-effect forces are given by

$$F_{\rm a}^{(w)} = -\frac{1}{2}\frac{\partial A^{(a)}}{\partial X_1}U^2, \quad F_{\rm b}^{(w)} = \frac{1}{2}\frac{\partial A^{(b)}}{\partial X_1}U^2, \quad X_1 = h. \tag{22}$$

The corresponding added-masses for these two cases can be readily determined from Milne-Thompson (1960, p. 504), as the leading-order terms for $(a/h) \ll 1$, i.e.,

$$A^{(\rm a)} = \frac{1}{2}\rho\forall\left[1 + \frac{3}{8}\left(\frac{a}{h}\right)^3\right], \quad A^{(\rm b)} = \frac{1}{2}\rho\forall\left[1 + \frac{3}{16}\left(\frac{a}{h}\right)^3\right], \tag{23}$$

which, when combined with (22), lead eventually to

$$F_a^{(w)} = \frac{9}{32}\frac{\rho\forall}{a}\left(\frac{a}{h}\right)^4 U^2; \quad F_b^{(w)} = -\frac{9}{64}\frac{\rho\forall}{a}\left(\frac{a}{h}\right)^4 U^2. \tag{24}$$

Clearly, these wall effect forces become significant as the bubble is in closed proximity to the wall, i.e., as $(a/h) \to 1^-$, and may be neglected altogether if the bubble is far from the wall. Hence, it is possible to find a critical distance from the wall, say h_c, for which the baroclinic and wall-induced forces, which act in opposite directions, precisely cancel each other resulting in a force-free motion. Comparing (21) and (24) implies that such a mutual cancellation occurs for

$$\frac{a\nabla\rho}{\rho}\left(\frac{h_c}{a}\right)^4 = \frac{9}{8}(\text{Case A}) = \frac{9}{16}(\text{Case B}) \tag{25}$$

One may conclude therefore that hydrodynamic wall effects in a weakly stratified flow can be ignored compared to the baroclinic effects if the distance from the wall is larger than $a(\rho/a\nabla\rho)^{1/4}$.

4. Pulsating bubble near a wall

In order to demonstrate the coupling effect between weak stratification and body surface deformation, let us consider the case of a stationary bubble lying at a distance h from a rigid wall in a quiescent fluid of uniform density gradient $\nabla\rho$. If the direction of gravity is colinear with X_1, then the bubble will be in perfect equilibrium providing its average density is equal to the fluid density evaluated at the bubble center. Now, let us assume that due to some forcing mechanism the bubble starts to pulsate harmonically about its mean radius a with an amplitude $a\epsilon$ and constant angular frequency ω, such that sphericity is preserved for all time (see Fig. 1, Case C). As a result of this non-isochoric pulsations the bubble will experience a Bjerknes type force of attraction towards the wall (see for example, Pelekasis and Tsamopoulos, 1993). Since here $U_\alpha = 0$ and the surface deformation is radially symmetric, the time derivative of the Kelvin-impulse in (7) for an homogeneous fluid can be ignored with respect to $(\partial A_{dd}/\partial X_1)$, when seeking the leading-order force to $o(a/h)$. In order to evaluate A_{dd}, we first note that the far-field (unbounded) deformation potential φ_d, can be described by a point source located at the bubble center with an output $a^3\omega$. By considering its first image in the rigid wall, one gets from (7) and (11) that to leading-order,

$$F_c^{(w)} = -2\pi\rho a^4\varepsilon^2\omega^2(a/h)^2 \tag{26}$$

confirming the well result that the Bjerknes force acting on two pulsating bubbles is of an attraction nature and is inversely proportional to the square of the distance between them.

Nevertheless, for a non-homogeneous flow, our theory predicts that there is yet another force component which arises from the coupling between the body's pure surface deformation and the uniform ambient density gradient, namely

$$F_c^{(\rho)} = A_{dd}\frac{\partial}{\partial X_1}(\ln\rho) = 4\pi a^5\varepsilon^2\omega^2|\nabla\rho| \tag{27}$$

which unlike $F_c^{(\rho)}$, does not depend (to leading-order) on the distance from the wall and in a similar manner to $F_b^{(\rho)}$, tends to push the bubble in the direction of the density gradient (i.e., away from the wall in this case). Thus, as in Cases A and B modeling rigid bubbles moving rectilinearly in the proximity of a wall, also in the case of a pulsating bubble, the baroclinic and inertia (wall-effect) forces, act in opposite directions. One can define here also a critical distance h_c, for which the inertia (deformation) and the baroclinic force terms balance each other, i.e., $h_c/a = (\rho/2a|\nabla\rho|)^{1/2}$. It can be shown that the analysis above is valid during a time interval satisfying $0 < \omega t < (\rho/a|\nabla\rho|)$.

5. Summary

An analytic study of the pressure induced forces and moments acting on a general rigid or deformable body moving unsteadily with six degrees of freedom in an otherwise quiescent bounded domain of an inviscid and incompressible fluid with a weak density gradient is presented. The analysis is based on using the generalized Kelvin–Kirchhoff equations of motion and the energy based added-mass formulation of Miloh and Landweber (1981), by taking into account the flow stratification and the proximity of interfaces and other rigid boundaries. In the case of rigid bodies embedded in an unbounded expanse of inviscid fluid, the general expressions reduce to those recently obtained by Eames and Hunt (1997) and Palierne (1999) and the equivalence between these two independent formulations (including the redundant distinction between 2-D and 3-D shapes) is also established.

It appears that if the rectilinear body velocity is perpendicular to the uniform density gradient the torque on the body is null regardless of its angular velocity. The body will experience a lift force, as long as the linear impulse is non-colinear with the density gradient. There exist an additional force acting in the direction of $\nabla\rho$, if the angular impulse is not orthogonal to the angular velocity. Thus, bearing the D'Alembert paradox in mind, it is demonstrated that a rigid body moving steadily in an unbounded inviscid fluid domain possessing weak uniform density gradient, generally exhibits a drag/thrust in a direction opposite to $\nabla\rho$ and a lift force in the direction of $\nabla\rho$.

The baroclinic reactions are to leading-order uncoupled from the viscous effects as well as from the common inertia terms resulting from the proximity of fluid interfaces and solid boundaries (i.e., wall effects) or those induced by body deformation or shape oscillations (i.e., Bjerknes forces). These baroclinic type pressure loads, which are usually most pronounced near flow boundaries, must be combined with the usual inertia and viscous effects acting on the particle in order to determine its physical trajectory. It should be also noted that while viscous effects may be ignored with respect to inertia especially in the case of clean bubbles accelerating in unbounded fluid, the proximity of rigid boundaries generally enhanced the shear induced viscous terms which can dominate the overall particle dynamics. However, the analysis of viscous wall effects is beyond the scope of this note, which is performed here within the framework of inviscid flow theory, in an attempt to isolate and capture the leading inertia and flow stratification induced effects.

The general analysis is demonstrated by considering a rather simple example of a rigid or pulsating spherical shape (bubble) moving near a rigid plane boundary in the presence of a uniform density gradient in a direction normal to the plane. Special attention is payed to the limiting and interesting cases where there is a perfect cancelation between the baroclinic and inertia effects.

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